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Semi-Progressions

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Let $g(n) \geqslant 0$ be a function. A sequence of k positive integers, $a_1 < a_2 < \cdots < a_k$, is called a k-term semi-progression for $g(n)$ provided the diameter of the set of differences, $\operatorname{diam}\{a_{j+1} - a_j \mid j = 1, 2, ..., k-1\}$, does not exceed $g(k)$. A set A of

similar to the earlier definition of having property QP (containing arbitrarily long quasi-progressions of bounded diameter.) For unbounded functions g the property SP(g) is quite new and this paper examines its relation to several other properties each of which is a generalization of the property AP of containing arbitrarily long arithmetic progressions.

1. INTRODUCTION

In [1] several generalizations of the property "AP" of a set of natural numbers containing arbitrarily long arithmetic progressions were considered. A familiarity with [1] is assumed. In the present paper another such generalization is presented and its place in the hierarchy of these properties is considered. Let $g(n)$ be an arbitrary non-negative function defined on the natural numbers. A sequence of k positive integers, $a_1 < a_2 < \cdots < a_k$, is called a k-term semi-progression for $g(n)$ (denoted $k - \mathrm{SP}(g)$) provided the diameter of the set of differences, $\{a_{j+1} - a_j \mid j = 1, 2, ..., k-1\}$, does not exceed $g(k)$. In other words, the sequence $\{a_j \mid 1 \leqslant j \leqslant k\}$ takes the form

$$
\begin{aligned}
a_2 &= a_1 + d + r_1 \\
a_3 &= a_2 + d + r_2 \\
&\vdots \\
a_k &= a_{k-1} + d + r_{k-1},
\end{aligned}
$$

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where d is a non-negative integer and each r_i satisfies $0 \leqslant r_i \leqslant g(k)$. Note that the defining condition for a semi-progression depends on the length, k, of the sequence. If $g(n)$ is a constant function with value d, then this definition coincides with that in [1] of being a k-term quasi-progression of diameter d, a $k-\mathrm{QP}(d)$. For a given $g(n)$, we will say that a set, A, of integers has property $\mathrm{SP}(g)$, if, for infinitely many k, A contains a $k-\mathrm{SP}(g)$. (It seems natural to require only infinitely many k here rather than all k or all large k. One reason is that a t-term subsequence of a $k-\mathrm{SP}(g)$ may not be a $t-\mathrm{SP}(g)$.) We note that our sequences $k-\mathrm{SP}(g)$ are similar to, but different from, those of Landman and Greenwell [3].

We observe that $\mathrm{AP} \Rightarrow \mathrm{SP}(g)$ for any g since the diameter of the difference set of an arithmetic progression is 0. The property, QP, of containing arbitrarily long quasi-progressions does not imply $\mathrm{SP}(g)$ if $g(n)$ is a bounded function. However, if $\lim g(n) = \infty$, then it is clear that $\mathrm{QP} \Rightarrow \mathrm{SP}(g)$. We will assume throughout that g is unbounded.

If $g(n)$ increases very rapidly then the property $\mathrm{SP}(g)$ may not even imply the weakest property considered in [1], namely, DW (containing arbitrarily long descending waves). For example, if $g(n) = 2^n$, then $A = \{2^i \mid i = 1, 2, 3, \ldots\}$ (which has no $3-\mathrm{DW}$) does have property $\mathrm{SP}(g)$ as 2, 4, 8, ..., 2^k is a $k-\mathrm{SP}(g)$. Normally, we will assume that $g(n)$ is bounded above by a polynomial. In this case we will show in Section 2 that $\mathrm{SP}(g) \Rightarrow \mathrm{DW}$.

Trivially, if $g(n) \leqslant h(n)$, then $\mathrm{SP}(g) \Rightarrow \mathrm{SP}(h)$. But, in general, the property $\mathrm{SP}(g)$ is quite sensitive to the function g. For example, the set of squares, $\{i^2 \mid i = 1, 2, 3, \ldots\}$, clearly possesses property $\mathrm{SP}(2n-4)$ since the sequence $1^2, 2^2, \ldots, k^2$ has largest difference of $k^2 - (k-1)^2 = 2k-1$ and smallest difference equal to 3. We will show in the next paragraph that the squares have property $\mathrm{SP}(2n-C)$ for any positive constant C. In a forthcoming paper we will prove the better result that the squares have property $\mathrm{SP}((1.5+\varepsilon)\,n)$. Whether or not the squares have property $\mathrm{SP}(n)$ seems a difficult question.

To prove that the set of squares has property $\mathrm{SP}(2n-C)$, we may proceed as follows: Let $k > j > 2$ and set

$$t = j + (j-1) + (j-2) + \cdots + 2 + 1 = \frac{j(j+1)}{2}, \qquad s = k - j.$$

Consider the following k-term progression of squares:

$$j^2, (j+(j-1))^2, (j+(j-1)+(j-2))^2, \ldots, t^2, (t+1)^2, (t+2)^2, \ldots, (t+s)^2.$$

If k is large with respect to j then the largest difference is $(t+s)^2 - (t+s-1)^2 = 2t + 2s - 1$. We show that the smallest difference is the

$(j-1)$th: $t^2-(t-1)^2=2t-1$. For this we need only look at differences among the first j terms. The rth difference is

$$\begin{aligned} d(r) &= (j+(j-1)+\cdots+(j-r+1)+(j-r))^2 \\ &\quad - (j+(j-1)+\cdots+(j-r+1))^2 \\ &= j(j+1)(j-r)-(j-r)^3. \end{aligned}$$

Differentiating twice with respect to r we obtain $d''(r)=-6(j-r)<0$. Thus the least value of $d(r)$ is either $d(1)=(j-1)(3j-1)$ or $d(j-1)=j(j+1)-1$ and the latter of these is smaller for $j\geqslant 3$. Hence the diameter of the difference set is $2s=2k-2j$ so that property SP$(2n-C)$ is satisfied if j is held fixed greater than $C/2$.

We conjecture that, whenever $g(n)$ is bounded above by a polynomial, SP$(g(n))\Rightarrow$ SP$(g(n)-C)$. But this is not true in general. For, taking A to be the set of powers of 2, we calculate that the k-term initial segment $1, 2, 4, \ldots, 2^{k-1}$ has difference set diameter $=2^{k-2}-1$ so that A has SP$(2^{k-2}-1)$. On the other hand, it is not hard to see that the diameter of the difference set in *any* k-progression, $2^{b_1}<2^{b_2}<\cdots<2^{b_{k-1}}<2^{b_k}$, in A is equal to

$$\begin{aligned} &2^{b_k}-2^{b_{k-1}}-(2^{b_2}-2^{b_1}) \\ &\quad \geqslant (2^{k+s}-2^{k+s-1})-(2^{s+2}-1) \qquad (s=b_k-k) \\ &\quad = 2^{k+s-1}-2^{s+2}+1 \\ &\quad = 2^{k-3}2^{s+2}-2^{s+2}+1 \\ &\quad = (2^{k-3}-1)\,2^{s+2}+1 \\ &\quad \geqslant 4(2^{k-3}-1)+1 \geqslant 2^{k-2}-1. \end{aligned}$$

Hence A does not have SP$(2^{k-2}-1-1)$.

Finally, we note, in connection with Erdös' conjecture ($\sum_{a\in A}(1/a)=\infty\Rightarrow A$ has AP), that $\sum_{a\in A}(1/a)=\infty\Rightarrow A$ has SP(g) for any g which satisfies $\sum_{n=1}^{\infty}(1/g(n))<\infty$ (just look at initial segments). Therefore, an interesting conjecture is: $\sum_{a\in A}(1/a)=\infty\Rightarrow A$ has SP(n).

2. HIERARCHY

In [1] was proved that AP $\Rightarrow$ QP $\Rightarrow$ CP $\Rightarrow$ C $\Rightarrow$ DW and that none of these implications is reversible. In this section we will show that, under suitable conditions, SP$(g(n))$ lies between QP and DW but is incomparable with CP and C. Precisely, we have already noted that QP $\Rightarrow$ SP(g), provided g is unbounded, and we will prove, under the assumption that $g(n) \leqslant p(n)$, where $p(n)$ is a polynomial, that $\mathrm{SP}(g) \Rightarrow \mathrm{DW}$, that this implication is not reversible, and that, for any $g(n)$, $\mathrm{CP} \not\Rightarrow \mathrm{SP}(g)$. Further, under the assumption that $g(n) \geqslant n$, we will show that $\mathrm{SP}(g) \not\Rightarrow \mathrm{C}$. It follows from these that $\mathrm{SP}(g) \not\Rightarrow \mathrm{CP}$, if $g(n) \geqslant n$, and $\mathrm{C} \not\Rightarrow \mathrm{SP}(g)$, for all $g(n)$.

We do not know, for small g, say for example, $g(n) = \log n$, whether or not $\mathrm{SP}(g) \Rightarrow \mathrm{CP}$ or C.

THEOREM 1. *If* $g(n) \leqslant p(n)$, *where* $p(n)$ *is a polynomial, then* $\mathrm{SP}(g) \Rightarrow \mathrm{DW}$.

Proof. Let A be a set of positive integers which has property $\mathrm{SP}(g)$. We show that A has property DW by considering two cases:

Case 1. For arbitrarily large k, A contains a $k-\mathrm{SP}(g)$, $a_1, a_2, ..., a_k$, where

$$\begin{aligned} a_2 &= a_1 + d + r_1 \\ a_3 &= a_2 + d + r_2 \\ &\vdots \\ a_k &= a_{k-1} + d + r_{k-1}, \end{aligned} \tag{1}$$

$0 \leqslant r_i \leqslant g(k)$, and $d \geqslant kg(k)$.

Let t be a given positive integer. We may assume $k \geqslant t(t+1)/2$. Define

$$w(j) = k - \frac{(j+1)(j+2)}{2} + 1 \qquad (j = 0, 1, ..., t-1).$$

Let

$$\begin{aligned} b_t &= \quad a_{w(0)} = a_k \\ b_{t-1} &= a_{w(1)} = a_{k-2} \\ &\vdots \\ b_{t-j} &= a_{w(j)} \\ &\vdots \\ b_1 &= \quad a_{w(t-1)}. \end{aligned}$$

Now $\{b_i\}$ is a $t-\mathrm{DW}$ in A as we now show that for $j = 1, 2, ..., t-2$,

$$b_{t-(j-1)} - b_{t-j} \leqslant b_{t-j} - b_{t-(j+1)}. \tag{2}$$

The left side of (2) is

$$\begin{aligned}
a_{w(j-1)} &- a_{w(j)} \\
&= (j+1)\,d + r_{w(j)} + r_{w(j)+1} + \cdots + r_{w(j)+j} \\
&\leqslant (j+1)\,d + (j+1)\,g(k) \\
&= (j+2)\,d - d + (j+1)\,g(k) \\
&< (j+2)\,d,
\end{aligned}$$

where the last step follows from $(j+1)\,g(k) \leqslant (t-1)\,g(k) < (t(t+1)/2)\,g(k) \leqslant kg(k) \leqslant d$. On the other hand, $(j+2)\,d \leqslant b_{t-j} - b_{t-(j+1)}$ since this difference is the sum of $j+2$ differences of the form $d + r_i$.

Before stating Case 2 we recall that it follows from Corollary 1 to Theorem 5 in [1] that, if the set A does not contain any $t-\mathrm{DW}$, then, for any $\varepsilon > 0$, $|A \cap \{n+1, n+2, ..., n+m\}| < m^{\varepsilon}$ for all n and all sufficiently large m.

Case 2. Suppose, for all large k for which there is a $k-\mathrm{SP}(g)$, $a_1, a_2, ..., a_k$, in A, that we have, in Eqs. (1), $d < kg(k)$.

We have, for these k,

$$\begin{aligned}
a_k &- a_1 + 1 \\
&= a_{k-1} + d + r_{k-1} - a_1 + 1 \\
&= a_{k-2} + 2d + r_{k-1} + r_{k-2} - a_1 + 1 \\
&\ \vdots \\
&= a_1 + (k-1)\,d + r_{k-1} + \cdots + r_1 - a_1 + 1 \\
&\leqslant (k-1)\,d + (k-1)\,g(k) + 1 \\
&< (k-1)\,kg(k) + (k-1)\,g(k) + 1 \\
&\leqslant (k-1)\,kp(k) + (k-1)\,p(k) + 1 \\
&< k^D,
\end{aligned}$$

for some constant $D > 1$. Thus $k > (a_k - a_1 + 1)^{1/D}$ and

$$\begin{aligned}
|A \cap \{&(a_1-1)+1, (a_1-1)+2, ..., (a_1-1)+(a_k-a_1+1)\}| \\
&\geqslant k > (a_k - a_1 + 1)^{1/D}.
\end{aligned}$$

If A does not contain long descending waves, then this last inequality contradicts the lemma stated just before this case.

THEOREM 2. *There exists a set A such that A has* DW, *but for no polynomial, $p(n)$, does A have* SP(p).

Proof. According to [1, p. 84], there is an infinite set $A=\{a_1, a_2, a_3, ...\}$ which possesses property DW and such that the set of differences $\{a_{i+1}-a_i \mid i=1, 2, ...\}$ is precisely the set of powers of 2 without repetition. It follows that if $b_1<b_2<\cdots<b_k$ is any k-term sequence in A, then the largest difference is at least 2^{k-2} and so the diameter of the difference set must be at least 2^{k-3}. The theorem follows easily.

THEOREM 3. *For any $g(n)$,* CP $\nRightarrow$ SP(g).

Proof. Let $p(n)$ be a strictly increasing function such that $g(n)\leqslant p(n)$. We may also assume that p increases quickly enough so that for all $t>1$, $p(t^6)>t^3p(t^3)$. It suffices to prove that CP $\nRightarrow$ SP(p). Let $H(n)$ be a function which satisfies the following growth conditions:

$$H(t+1)>2(H(t)+tp(t^6)+tp(t^3))$$
$$H([t^{1/3}])>p(t).$$

We freely borrow ideas from the proof that CP $\nRightarrow$ QP in [1, pp. 84–85]. In particular we use J. Justin's sequence, $\{z_i\}$, of zeros and ones which does not contain five consecutive blocks of equal composition [2].

For $t\geqslant 1$, let

$$B(t)=\left\{H(t)+jp(t^6)+\left(\sum_{i=1}^{j} z_i\right)p(t^3) \mid j=1, 2, ..., t\right\}.$$

The jth difference of this t-term progression is equal to $p(t^6)$ or $p(t^6)+p(t^3)$, depending on whether $z_{j+1}=0$ or 1, respectively. Hence, for each t, $B(t)$ is a $t-\mathrm{CP}(2)$ so that the set $A=\bigcup_{t=1}^{\infty} B(t)$ has property CP. We will show that A does not have property SP(p).

Let $t(k)$ be an integer such that, for all large k, $k^{1/3}<t(k)<2k^{1/3}$. Now assume that, for infinitely many k, there is a $k-\mathrm{SP}(p)$ in A, say $G=\{b_1, b_2, ..., b_k\}$. We first show that, for sufficiently large k, G cannot intersect three distinct $B(t)$, where $t>t(k)$. Let $t(k)<t_1<t_2<t_3$ and suppose that $B(t_i)\cap G\neq\varnothing$ $(i=1, 2, 3)$. Set $b_r=\max(G\cap B(t_2))$ and $b_s\in G\cap B(t_1)$. Note that $b_{r+1}\in G\cap B(q)$, where $q\geqslant t_2+1$. The diameter of the difference set of G is

$$\begin{aligned}&\geqslant(b_{r+1}-b_r)-(b_{s+1}-b_s)\geqslant(b_{r+1}-b_r)-(b_r-b_s)\\&=b_{r+1}-2b_r+b_s>H(t_2+1)-2(H(t_2)+t_2p(t_2^6)+t_2p(t_2^3))+H(t_1)\\&>H(t_1)>H(t(k))>H([k^{1/3}])>p(k).\end{aligned}$$

This contradicts G being a $k-\mathrm{SP}(p)$.

Now $|B(1) \cup B(2) \cup \cdots \cup B(t(k))| = t(k)(t(k)+1)/2 \leqslant (t(k))^2 \leqslant 4k^{2/3}$. Since $|G| = k$, which, for large k, is much greater than $4k^{2/3}$, it follows that six consecutive terms of G, say $b_i, b_{i+1}, ..., b_{i+5}$, are contained in some $B(t)$, where $t > t(k)$. Consider two consecutive differences among these six terms. We have for suitable $u > v > w$ (where $t \geqslant u$)

$$\begin{aligned}
p(k) &\geqslant |b_{j+2} - 2b_{j+1} + b_j| \\
&= |H(t) + up(t^6) + (z_1 + \cdots + z_u)\, p(t^3) \\
&\quad - 2(H(t) + vp(t^6) + (z_1 + \cdots + z_v)\, p(t^3)) \\
&\quad + H(t) + wp(t^6) + (z_1 + \cdots + z_w)\, p(t^3)| \\
&= |(u + w - 2v)\, p(t^6) + ((z_{v+1} + \cdots + z_u) - (z_{w+1} + \cdots + z_v))\, p(t^3)|.
\end{aligned}$$

Now $k < (t(k))^3 < t^3$ so that $p(k) < p(t^3)$. Hence,

$$1 > \left| (u + w - 2v) \frac{p(t^6)}{p(t^3)} + ((z_{v+1} + \cdots + z_u) - (z_{w+1} + \cdots + z_v)) \right|$$

and it follows, since $p(t^6)/p(t^3) > t^3$ and both $|(u + w - 2v)|$ and $|(z_{v+1} + \cdots + z_u) - (z_{w+1} + \cdots + z_v)|$ are less than t, that these last two numbers are both zero. Thus, as in [1], Justin's sequence must contain five consecutive blocks of equal composition. This contradiction completes the proof.

THEOREM 4. *If g satisfies $g(n) \geqslant n$, then* $\mathrm{SP}(g) \nRightarrow C$.

Proof. It is sufficient to prove that $\mathrm{SP}(n) \nRightarrow C$. Let d_1 be a positive integer and

$$d_m > 2(m-1)\, d_{m-1} + m(m+1) \qquad \text{for} \quad m = 2, 3, 4, ... \tag{3}$$

For $m = 1, 2, ...,$ let $S_m = \{d_m + 1, 2d_m + 3, ..., md_m + m(m+1)/2\}$. Set $S = \bigcup_{m=1}^{\infty} S_m$. For each $k \geqslant 1$ the set S_k is a $k-\mathrm{SP}(n)$ since $(id_k + i(i+1)/2) - ((i-1)\, d_k + (i-1)\, i/2) = d_k + i$, where $2 \leqslant i \leqslant k$. Thus S has property $\mathrm{SP}(n)$.

We proceed to show that S does not have property C. In fact, we show that S contains no 2-cube. Assume to the contrary that $[a_0, y_1, y_2]$ is a 2-cube in S. Write

$$\begin{aligned}
a_0 &= \beta d_i + (\beta(\beta+1))/2, &\qquad 1 \leqslant \beta \leqslant i, \\
a_0 + y_1 &= \alpha d_j + (\alpha(\alpha+1))/2, &\qquad 1 \leqslant \alpha \leqslant j, \\
a_0 + y_2 &= \delta d_q + (\delta(\delta+1))/2, &\qquad 1 \leqslant \delta \leqslant q, \\
a_0 + y_1 + y_2 &= \gamma d_p + (\gamma(\gamma+1))/2, &\qquad 1 \leqslant \gamma \leqslant p.
\end{aligned}$$

Clearly $j \geqslant i$, $p \geqslant q$, and $p \geqslant j$. We also note that if $j = i$, then $\alpha > \beta$; if $p = q$, then $\gamma > \delta$; and if $p = j$, then $\gamma > \alpha$.

If $p > j$, we must have $p > q$ for otherwise we have

$$\begin{aligned}
0 &= (a_0 + y_1 + y_2) - (a_0 + y_2) - (a_0 + y_1) + a_0 \\
&= \gamma d_p + \frac{\gamma(\gamma+1)}{2} - \delta d_q - \frac{\delta(\delta+1)}{2} - \alpha d_j - \frac{\alpha(\alpha+1)}{2} + \beta d_i + \frac{\beta(\beta+1)}{2} \\
&= (\gamma - \delta)\, d_p - \alpha d_j - \frac{\alpha(\alpha+1)}{2} + \left(\frac{\gamma(\gamma+1)}{2} - \frac{\delta(\delta+1)}{2} + \beta d_i + \frac{\beta(\beta+1)}{2}\right) \\
&\geqslant d_p - j d_j - \frac{j(j+1)}{2} \geqslant d_p - (p-1)\, d_{p-1} - \frac{(p-1)\,p}{2} > 0.
\end{aligned}$$

The last step follows from (3).

Suppose now $p > j$ and $p > q$. Similarly, we have

$$\begin{aligned}
0 &= (a_0 + y_1 + y_2) - (a_0 + y_2) - (a_0 + y_1) + a_0 \\
&= \gamma d_p + \frac{\gamma(\gamma+1)}{2} - \delta d_q - \frac{\delta(\delta+1)}{2} - \alpha d_j - \frac{\alpha(\alpha+1)}{2} + \beta d_i + \frac{\beta(\beta+1)}{2} \\
&> d_p - q d_q - \frac{q(q+1)}{2} - j d_j - \frac{j(j+1)}{2} \\
&\geqslant d_p - (p-1)\, d_{p-1} - \frac{(p-1)\,p}{2} - (p-1)\, d_{p-1} - \frac{(p-1)\,p}{2} \\
&= d_p - 2(p-1)\, d_{p-1} - (p-1)\, p > 0.
\end{aligned}$$

Hence, we have $p = j$. Now assume $p > q$. Again,

$$\begin{aligned}
0 &= (a_0 + y_1 + y_2) - (a_0 + y_2) - (a_0 + y_1) + a_0 \\
&= \gamma d_p + \frac{\gamma(\gamma+1)}{2} - \delta d_q - \frac{\delta(\delta+1)}{2} - \alpha d_j - \frac{\alpha(\alpha+1)}{2} + \beta d_i + \frac{\beta(\beta+1)}{2} \\
&> (\gamma - \alpha)\, d_p - (p-1)\, d_{p-1} - \frac{(p-1)\,p}{2} > 0.
\end{aligned}$$

Therefore, $p = q = j$, $\alpha < \gamma$, and $\delta < \gamma$. Now assume $i < j$. First

$$\gamma d_p + \frac{\gamma(\gamma+1)}{2} - \delta d_q - \frac{\delta(\delta+1)}{2} = y_1 = \alpha d_j + \frac{\alpha(\alpha+1)}{2} - \beta d_i - \frac{\beta(\beta+1)}{2}.$$

Thus,

$$\beta d_i + \frac{\beta(\beta+1)}{2} = (\alpha+\delta-\gamma)\, d_p + \frac{1}{2}(\alpha(\alpha+1)+\delta(\delta+1)-\gamma(\gamma+1)). \qquad (4)$$

If $\alpha+\delta-\gamma \leqslant 0$, then $\alpha(\alpha+1)+\delta(\delta+1)-\gamma(\gamma+1)<0$ and so $a_0<0$, which is impossible. If $\alpha+\delta-\gamma>0$, then the RHS of (4) $\geqslant d_p - p(p+1)/2 > 2(p-1)\, d_{p-1} + p(p+1)/2$, while the LHS of (4) $\leqslant (p-1)\, d_{p-1} + p(p+1)/2$, which leads to a contradiction.

Hence, $i=j=p=q$. From $(a_0+y_1+y_2)-(a_0+y_2)=y_1=(a_0+y_1)-(a_0)$ we get

$$(\gamma-\delta)\, d_p + \frac{\gamma(\gamma+1)}{2} - \frac{\delta(\delta+1)}{2} = (\alpha-\beta)\, d_p + \frac{\alpha(\alpha+1)}{2} - \frac{\beta(\beta+1)}{2}. \qquad (5)$$

If $\gamma-\delta>\alpha-\beta$, then

$$((\gamma-\delta)-(\alpha-\beta))\, d_p = \frac{\alpha(\alpha+1)}{2} - \frac{\beta(\beta+1)}{2} - \frac{\gamma(\gamma+1)}{2} + \frac{\delta(\delta+1)}{2}. \qquad (6)$$

Here the LHS of (6) $\geqslant d_p > p(p+1)$ and the RHS of (6) $\leqslant p(p+1)$, a contradiction. A similar contradiction is obtained when $\gamma-\delta<\alpha-\beta$. Hence, $\gamma-\delta=\alpha-\beta$.

Now (5) becomes $\gamma(\gamma+1)-\delta(\delta+1)=\alpha(\alpha+1)-\beta(\beta+1)$ which reduces to $\gamma^2-\delta^2=\alpha^2-\beta^2$, or $(\gamma+\delta)(\gamma-\delta)=(\alpha+\beta)(\alpha-\beta)$. If $\gamma-\delta=0$, then

$$y_1 = \gamma d_p + \frac{\gamma(\gamma+1)}{2} - \delta d_p - \frac{\delta(\delta+1)}{2} = 0,$$

a contradiction. Hence, $\gamma-\delta\neq 0$, so that $\gamma+\delta=\alpha+\beta$. Thus, finally, we obtain $\gamma=\alpha$ and $\delta=\beta$ which is the contradiction that concludes the proof.

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